Generalized stochastic Schrödinger equations for state vector collapse

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Abstract

A number of authors have proposed stochastic versions of the Schrödinger equation, either as effective evolution equations for open quantum systems or as alternative theories with an intrinsic collapse mechanism. We discuss here two directions for generalization of these equations. First, we study a general class of norm preserving stochastic evolution equations, and show that even after making several specializations, there is an infinity of possible stochastic Schrödinger equations for which state vector collapse is provable. Second, we explore the problem of formulating a relativistic stochastic Schrödinger equation, using a manifestly covariant equation for a quantum field system based on the interaction picture of Tomonaga and Schwinger. The stochastic noise term in this equation can couple to any local scalar density that commutes with the interaction energy density, and leads to collapse onto spatially localized eigenstates. However, as found in a similar model by Pearle, the equation predicts an infinite rate of energy nonconservation proportional to $\delta^3(\vec{0})$, arising from the local double commutator in the drift term.

I. Introduction

The measurement problem is widely perceived as the greatest difficulty in the interpretation of quantum mechanics: how, without invoking a separate realm of classical measuring devices, can one rule out superpositions of macroscopically distinct states, as in the famous Schrödinger's cat paradox? To answer this question, a number of authors have suggested modifying the usual Schrödinger equation so as to eliminate such superpositions at large length scales, while retaining the standard quantum results for microscopic systems. The result is a modified Schrödinger equation containing extra terms, including *stochastic* terms which reproduce the probabilities of measurements [1–7].

In a parallel development, other researchers have derived effective equations to describe systems evolving in contact with an external environment. These effective equations also take the form of stochastic Schrödinger equations, of a form very similar to those posited in response to the measurement problem [8–14].

One example of such a modified equation is the *quantum state diffusion* (QSD) equation of Gisin and Percival [12], which has the form

$$|d\psi\rangle = -i\hat{H}|\psi\rangle dt + \sum_k \left(\langle \hat{L}_k^\dagger \rangle \hat{L}_k - \frac{1}{2}\hat{L}_k^\dagger \hat{L}_k - \frac{1}{2}|\langle \hat{L}_k \rangle|^2 \right) |\psi\rangle dt + \sum_k (\hat{L}_k - \langle \hat{L}_k \rangle)|\psi\rangle d\xi_k \quad . \quad (1)$$

Here the *Lindblad operators* $\hat{L}_k$ [15] represent the effects of the environment, $\hat{H}$ is the Hamiltonian, and the stochastic differentials $d\xi_k$ represent independent complex Wiener processes with vanishing ensemble averages or means (i.e., $M[d\xi_k] = 0$), that obey the Itô stochastic calculus

$$d\xi_j^* d\xi_k = dt\delta_{jk} \quad , \quad d\xi_j d\xi_k = dt d\xi_k = 0 \quad . \quad (2)$$

Equations (1) and (2) define an Itô stochastic differential equation; in manipulations using the Itô differential d, one must use the modified chain rule $d(AB) = dAB + AdB + dAdB$. While the dynamics of $|\psi\rangle$ can be extremely complex, there is a tendency for the state to *localize* onto eigenstates of the Lindblad operators $\hat{L}_k$. Of course, the competing influences of different $\hat{L}_k$, or of the Hamiltonian $\hat{H}$, can prevent this localization from taking place. Note also that Eq. (1) is nonlinear in $|\psi\rangle$; this will in general be necessary for such an equation to preserve the norm of the state.

While we have presented this as an effective equation, arising due to the effects of an external environment, one can postulate an exactly similar equation in which the noise is considered fundamental. Percival has proposed such an equation with localization onto energy eigenstates, which he calls *Primary* state diffusion (or PSD) [16]. Other such equations have been proposed by Pearle, by Ghirardi, Rimini and Weber, by Diósi, by Ghirardi, Pearle, and Rimini, and by Hughston [1,3,5,7]. A survey of their properties has recently been given by Adler and Horwitz [17], who give a detailed discussion of the conditions for the dynamics of Eq. (1) to lead to state vector collapse.

Our aim in this paper is twofold. First, we examine the extent to which a stochastic dynamics such as Eq. (1) can be kept in its most general form, subject to the requirement that it should still lead to state vector collapse. This forms the subject matter of Sec. II, where we show that there is an infinite parameter family of stochastic equations for which state vector collapse is provable.

Our second aim is to explore the well-known problem that all equations with the structure of Eq. (1) are nonrelativistic. They are designed to mimic measurement, and they almost all contain a distinguished frame which takes the role of the rest frame of the measuring device. Since in standard QM measurements take effect instantaneously on the state vector of the entire system—ultimately, on the entire universe—it has been very difficult to find a covariant theory of measurement. In Sec. III we study a local generalization of Eq. (1) which can be written in manifestly covariant form, based on the "many fingered time" Tomonaga-Schwinger generalization of the Schrödinger equation. (For previous related approaches to this problem, see e.g. [18–21].) The generalized equation, like its nonrelativistic counterparts, causes the values of certain quantities (such as the center of mass of a measuring meter) to localize. However, there are difficulties with energy conservation arising from the local structure of the stochastic terms.

II. Generalized Stochastic Equations

II.1 General Framework

We begin by giving a general framework for the basic QSD equation of Eq. (1). Consider the stochastic state evolution

$$|d\psi\rangle = \hat{\alpha}|\psi\rangle dt + \sum_k \hat{\beta}_k|\psi\rangle d\xi_k \quad , \tag{3}$$

with $d\xi_k$ independent complex Wiener processes as in Eqs. (1) and (2), and with $\hat{\alpha}$ and $\hat{\beta}_k$ the operator coefficients of the drift and stochastic terms respectively, which can also have an explicit dependence on the state $|\psi\rangle$. The condition for this evolution to be stochastic unitary, so that the norm of the state is preserved, is

$$0 = d\langle\psi|\psi\rangle = \langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle + \langle d\psi|d\psi\rangle \quad . \tag{4}$$

Substituting Eq. (3) and its adjoint, and using Eq. (2) to simplify the quadratic terms in the Itô differentials, this becomes

$$0 = dt\langle\psi|\hat{\alpha} + \hat{\alpha}^\dagger + \sum_k \hat{\beta}_k^\dagger\hat{\beta}_k|\psi\rangle + \sum_k [d\xi_k^*\langle\psi|\hat{\beta}_k^\dagger|\psi\rangle + d\xi_k\langle\psi|\hat{\beta}_k|\psi\rangle] \quad . \tag{5}$$

Since $d\xi_k$ and $d\xi_k^*$ are independent, Eq. (5) requires that the coefficients of dt, $d\xi_k$, and $d\xi_k^*$ vanish independently, giving the conditions

$$\begin{aligned} 0 =& \langle\psi| \left(\hat{\alpha} + \hat{\alpha}^\dagger + \sum_k \hat{\beta}_k^\dagger \hat{\beta}_k \right) |\psi\rangle \quad , \\ 0 =& \langle\psi|\hat{\beta}_k|\psi\rangle \quad , \quad \text{all} \quad k \quad . \end{aligned} \tag{6}$$

Letting $\hat{L}_k$ be a set of general (not necessarily self-adjoint) operators, and $\hat{H} = \hat{H}^\dagger$ and $\hat{K} = \hat{K}^\dagger$ be arbitrary self-adjoint operators, the general solution to the conditions of Eq. (6) takes the form

$$\begin{aligned} \hat{\beta}_k =& \hat{L}_k - \langle\psi|\hat{L}_k|\psi\rangle \quad , \\ \hat{\alpha} =& - i\hat{H} + \hat{K} - \langle\psi|\hat{K}|\psi\rangle - \frac{1}{2}\sum_k \hat{\beta}_k^\dagger \hat{\beta}_k \quad , \end{aligned} \tag{7}$$

with the operators $\hat{K}$ and $\hat{L}_k$ still allowed to have an explicit dependence on the state vector $|\psi\rangle$. It is convenient for what follows to introduce the definitions

$$\begin{aligned} \langle\hat{O}\rangle \equiv& \langle\psi|\hat{O}|\psi\rangle \quad , \\ \Delta\hat{O} \equiv& \hat{O} - \langle\hat{O}\rangle \quad , \end{aligned} \tag{8}$$

where $\hat{O}$ is an arbitrary operator. Then Eq. (7) can be written in somewhat more compact form as

$$\begin{aligned} \hat{\beta}_k =& \Delta\hat{L}_k \quad , \\ \hat{\alpha} =& - i\hat{H} + \Delta\hat{K} - \frac{1}{2}\sum_k \hat{\beta}_k^\dagger \hat{\beta}_k \quad . \end{aligned} \tag{9}$$

Equations (3) and (7)-(9) give the general form of a norm-preserving stochastic extension of the Schrödinger equation. Equation (1) clearly has this general form, with the specific choice $\hat{K} = \frac{1}{2}\sum_k(\langle\hat{L}_k^\dagger\rangle\hat{L}_k - \hat{L}_k^\dagger\langle\hat{L}_k\rangle)$, for which $\langle\hat{K}\rangle = 0$ so $\Delta\hat{K} = \hat{K}$. Usually, in applications of the QSD equation it is assumed that the Lindblads have no dependence on the state $|\psi\rangle$, but we will find it useful to keep open the possibility that they do have a nontrivial state dependence.

To analyze convergence properties implied by this equation, we shall need formulas for the evolution of the expectation $\langle \hat{O} \rangle$ and the variance $V[\hat{O}] \equiv \langle (\Delta \hat{O})^2 \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$ of a general operator $\hat{O}$. Using Eq. (2) and the Itô extension of the chain rule, together with Eq. (3) and its adjoint, and (in the calculation of dV) imposing the normalization constraints of Eq. (9), we find after some algebra the results

$$
\begin{aligned}
d\langle \hat{O} \rangle =& \langle \frac{d\hat{O}}{dt} + \hat{\alpha}^\dagger \hat{O} + \hat{O}\hat{\alpha} + \sum_k \hat{\beta}_k^\dagger \hat{O} \hat{\beta}_k \rangle dt \\
&+ \sum_k [d\xi_k \langle \hat{O}\hat{\beta}_k \rangle + d\xi_k^* \langle \hat{\beta}_k^\dagger \hat{O} \rangle] \quad , \\
dV[\hat{O}] =& [\langle \Delta \hat{O} \frac{d\hat{O}}{dt} + \frac{d\hat{O}}{dt} \Delta \hat{O} \rangle \\
&+ \langle \hat{\alpha}^\dagger (\Delta \hat{O})^2 + (\Delta \hat{O})^2 \hat{\alpha} + \sum_k \hat{\beta}_k^\dagger (\Delta \hat{O})^2 \hat{\beta}_k \rangle - 2 \sum_k \langle \hat{\beta}_k^\dagger \Delta \hat{O} \rangle \langle \Delta \hat{O} \hat{\beta}_k \rangle] dt \\
&+ \sum_k [d\xi_k \langle (\Delta \hat{O})^2 \hat{\beta}_k \rangle + d\xi_k^* \langle \hat{\beta}_k^\dagger (\Delta \hat{O})^2 \rangle] \quad .
\end{aligned}
\tag{10}
$$

In applying Eq. (10), we shall have occasion to take its mean over the Itô process. Since the stochastic expectation or Itô process mean $M[\ ]$ obeys

$$
M[d\xi_k S] = M[d\xi_k^* S] = 0 \quad , \quad \text{all} \quad k \quad , \tag{11}
$$

for a general Hilbert space scalar S, the terms in Eq. (10) involving $d\xi_k$ and $d\xi_k^*$ drop out in the mean, giving

$$
\begin{aligned}
M[d\langle \hat{O} \rangle] =& M[\langle \frac{d\hat{O}}{dt} + \hat{\alpha}^\dagger \hat{O} + \hat{O}\hat{\alpha} + \sum_k \hat{\beta}_k^\dagger \hat{O} \hat{\beta}_k \rangle] dt \quad , \\
M[dV[\hat{O}]] =& M[\langle \Delta \hat{O} \frac{d\hat{O}}{dt} + \frac{d\hat{O}}{dt} \Delta \hat{O} \rangle \\
&+ \langle \hat{\alpha}^\dagger (\Delta \hat{O})^2 + (\Delta \hat{O})^2 \hat{\alpha} + \sum_k \hat{\beta}_k^\dagger (\Delta \hat{O})^2 \hat{\beta}_k \rangle - 2 \sum_k \langle \hat{\beta}_k^\dagger \Delta \hat{O} \rangle \langle \Delta \hat{O} \hat{\beta}_k \rangle] dt \quad .
\end{aligned}
\tag{12}
$$

Clearly, these equation take the same form if $\hat{O}$ is replaced everywhere by any function $\hat{F}[\hat{O}]$, since this simply defines a new operator $\hat{F}$ that replaces the dummy operator $\hat{O}$. In the next two sections we shall argue that for the evolution given by Eq. (3) to converge to an eigenstate of $\hat{O}$, we must have $M[dV[\hat{O}]] \leq 0$, with equality only for $\langle(\Delta\hat{O})^2\rangle = 0$, and shall demonstrate this for a particular special class of equations.

II.2 Specialization

We shall now introduce some simplifying specializations, which as we shall see, still leave an infinite parameter class of stochastic Schrödinger equations, for which state vector reduction to eigenstates of the operator $\hat{O}$ is provable. First of all, let us restrict ourselves to the case in which $\hat{O}$ is a self-adjoint observable, which we assume to have no explicit time dependence, so that $\hat{O} = \hat{O}^\dagger \quad , \quad d\hat{O}/dt = 0$. Secondly, let us specialize the choice of the operators $\hat{\alpha}$ and $\hat{\beta}_k$, which satisfy the normalization constraints of Eq. (9), as follows:

(i) We take the operator $\hat{K}$ to be zero, so that the constraint of Eq. (6) is satisfied as an operator relation

$$\hat{\alpha} + \hat{\alpha}^\dagger + \sum_k \hat{\beta}_k^\dagger \hat{\beta}_k = 0 \quad , \tag{13}$$

which as in Eq. (9) implies that

$$\hat{\alpha} = -i\hat{H} - \frac{1}{2}\sum_k \hat{\beta}_k^\dagger \hat{\beta}_k \quad . \tag{14}$$

(ii) We take $\hat{H}$ to be an operator that commutes with $\hat{O}$, and take all of the $\hat{L}_k$ to be functions solely of the operator $\hat{O}$, so that they also commute with $\hat{O}$,

$$[\hat{H},\hat{O}]=0 \quad , \quad \hat{L}_k \equiv \hat{L}_k[\hat{O}] \Rightarrow [\hat{L}_k,\hat{O}]=0 \quad . \tag{15}$$

Together with Eqs. (7) and (13), these specializations imply that $\hat{\alpha}$ and $\hat{\beta}_k$ all commute with $\hat{O}$, as well as with any function $\hat{F}[\hat{O}]$ solely of the operator $\hat{O}$,

$$\begin{aligned} [\hat{\alpha},\hat{O}]=0 \quad , \quad [\hat{\beta}_k,\hat{O}]=0 \quad , \\ [\hat{\alpha},\hat{F}[\hat{O}]]=0 \quad , \quad [\hat{\beta}_k,\hat{F}[\hat{O}]]=0 \quad , \end{aligned} \tag{16}$$

With these specializations, Eqs. (12) for the time derivatives of the stochastic mean of the quantum expectation of a function $\hat{F}[\hat{O}]$, and of the stochastic mean of the variance of $\hat{O}$, simplify dramatically. Since $\hat{\alpha}$ and $\hat{\beta}_k$ commute with $\hat{O}$, as well as with any function $\hat{F}[\hat{O}]$, we have

$$\begin{aligned} \langle \hat{\alpha}^\dagger(\Delta\hat{O})^2+(\Delta\hat{O})^2\hat{\alpha}+\sum_k \hat{\beta}_k^\dagger(\Delta\hat{O})^2\hat{\beta}_k\rangle =&\langle(\Delta\hat{O})^2[\hat{\alpha}+\hat{\alpha}^\dagger+\sum_k \hat{\beta}_k^\dagger\hat{\beta}_k]\rangle=0 \quad , \\ \langle \hat{\alpha}^\dagger\hat{F}[\hat{O}]+\hat{F}[\hat{O}]\hat{\alpha}+\sum_k \hat{\beta}_k^\dagger\hat{F}[\hat{O}]\hat{\beta}_k\rangle =&\langle\hat{F}[\hat{O}][\hat{\alpha}+\hat{\alpha}^\dagger+\sum_k \hat{\beta}_k^\dagger\hat{\beta}_k]\rangle=0 \quad , \end{aligned} \tag{17}$$

where we have used the operator constraint of Eq. (13). Also, since $\langle\Delta\hat{O}\rangle=0$, we have

$$\langle\Delta\hat{O}\hat{\beta}_k\rangle=\langle\Delta\hat{O}(\hat{L}_k-\langle\hat{L}_k\rangle)\rangle=\langle\Delta\hat{O}\hat{L}_k\rangle \quad , \tag{18}$$

and when $\hat{O}$ is self-adjoint, we have $\langle\hat{\beta}_k^\dagger\Delta\hat{O}\rangle=\langle\Delta\hat{O}\hat{\beta}_k\rangle^*$. Thus, what remains of Eq. (12) is

$$\begin{aligned} M[d\langle\hat{F}[\hat{O}]\rangle]=&0 \quad , \\ M[dV[\hat{O}]]=&-2M[\sum_k|\langle\Delta\hat{O}\hat{L}_k[\hat{O}]\rangle|^2]dt \quad , \end{aligned} \tag{19}$$

with $\hat{F}[\hat{O}]$ any function solely of the operator $\hat{O}$.

II.3 State Vector Reduction

We shall now show that the stochastic dynamics, as specialized in the preceding subsection, implies state vector reduction to eigenstates of $\hat{O}$ (assumed nondegenerate), with probabilities given by the Born rule in terms of the initial wave function. We shall need one further assumption beyond those introduced above, namely, that the scalar valued function f of $\hat{O}$ defined by

$$f[\hat{O}] \equiv \sum_k |\langle \Delta\hat{O}\hat{L}_k[\hat{O}]\rangle|^2 \tag{20}$$

vanishes if and only if $\langle(\Delta\hat{O})^2\rangle$ vanishes. One simple way to achieve this is to take $\hat{L}_k[\hat{O}]$ to have the form

$$\hat{L}_k[\hat{O}] = \sum_{n=0}^{N} c_k^{(n)} (\Delta\hat{O})^{2n+1} \quad , \tag{21a}$$

with $c_k^{(0)} > 0$ for at least one value of k; note that here we are using the freedom, remarked on above, to allow the Lindblads to have an explicit dependence on the state vector. This implies that for this value of k,

$$\langle \Delta\hat{O}\hat{L}_k[\hat{O}]\rangle = \sum_{n=0}^{N} c_k^{(n)} \langle[(\Delta\hat{O})^2]^{n+1}\rangle > c_k^{(0)}\langle(\Delta\hat{O})^2\rangle \quad , \tag{21b}$$

and so the vanishing of $f[\hat{O}]$ implies the vanishing of $\langle(\Delta\hat{O})^2\rangle$. This still leaves an infinite parameter freedom in the construction of the $\hat{L}_k$. A second specific example of a $f[\hat{O}]$ with the needed property is given in Sec. III.3 below.

A general condition for $f[\hat{O}]$ to have the needed property can be formulated by rewriting Eq. (20) as

$$f[\hat{O}] = \langle\psi|\Delta\hat{O}\hat{P}[\hat{O}]\Delta\hat{O}|\psi\rangle \quad , \quad \hat{P}[\hat{O}] \equiv \sum_k \hat{L}_k[\hat{O}]|\psi\rangle\langle\psi|\hat{L}_k^\dagger[\hat{O}] \quad , \tag{22}$$

with $\hat{P}[\hat{O}]$ by construction a positive semidefinite operator. If the Lindblads $\hat{L}_k$ were all unity, $\hat{P}$ would be proportional to the projector $|\psi\rangle\langle\psi|$, and since $|\psi\rangle$ is orthogonal to the state $\Delta\hat{O}|\psi\rangle$, one would have $f[\hat{O}] \equiv 0$. In order for $f[\hat{O}]$ to have the needed property, it is necessary for the Lindblads to introduce enough distortion of the projector $|\psi\rangle\langle\psi|$ for $\hat{P}[\hat{O}]$

to make a strictly positive contribution to Eq. (22), in which case the vanishing of $f[\hat{O}]$ requires the vanishing of the state $\Delta\hat{O}|\psi\rangle$, or equivalently, the vanishing of $\langle(\Delta\hat{O})^2\rangle$. This formulation of the condition on $f[\hat{O}]$ suggests that in the generic case, it is natural for it to have the needed property.

We can now proceed with a convergence proof, following the presentation given by Adler and Horwitz [17] (see also [1,4,7]). Integrating the second line of Eq. (19) with respect to t, we get

$$M[V[\hat{O}]](t) = M[V[\hat{O}]](0) - 2\int_0^t M[f[\hat{O}]](t)dt \quad . \tag{23}$$

Since both V and f are nonnegative, Eq. (23) implies that the integrand $M[f[\hat{O}]](t)$ must vanish as $t \to \infty$, since otherwise the right hand side of Eq. (23) would become negative at large times. This in turn implies that $f[\hat{O}](t)$ vanishes as $t \to \infty$ except on a set of probability measure zero, which by the assumption introduced following Eq. (20) implies that the variance $V[\hat{O}] = \langle(\Delta\hat{O})^2\rangle$ vanishes as $t \to \infty$ except on a set of probability measure zero. Thus, when $\hat{O}$ is nondegenerate, the state vector reduces to a pure state. Now integrating the first line of Eq. (19) with respect to t, taking the function $\hat{F}[\hat{O}]$ to be a projector Π_ℓ on the ℓth eigenstate of $\hat{O}$, we get

$$M[\langle\Pi_\ell(\infty)\rangle] = M[\langle\Pi_\ell(0)\rangle] = \langle\Pi_\ell(0)\rangle \quad . \tag{24}$$

The left hand side of Eq. (24) is just the probability that the stochastic process settles at $t = \infty$ on the ℓth eigenstate of $\hat{O}$, while the right hand side of Eq. (24) is the probability amplitude squared for the ℓth eigenstate to occur in the initial state vector $|\psi\rangle$. Thus, as first proposed by Pearle [1], the first line of Eq. (19)—which states that the stochastic process for $\langle\hat{F}[\hat{O}]\rangle$ is a martingale—implies that the state vector reduction implied by the second line of Eq. (19) obeys the Born probability rule.

To compare what we have done to the analyses of Hughston and of Adler and Horwitz [7,17], those authors consider the energy-driven case in which the operator $\hat{O} = \hat{H}$, and in which one (at least) of the $\Delta\hat{L}_k$ is simply taken as $\Delta\hat{H}$, corresponding to the case where the sum in Eq. (21) consists only of the $n = 0$ term. Note that in this case, it makes no difference whether we take $\hat{L}_k = \hat{H}$ or we take $\hat{L}_k = \Delta\hat{H}$, since either gives $\Delta\hat{L}_k = \Delta\hat{H}$. When $n > 0$ terms are present in $\hat{L}_k$, this distinction is important, and plays a role in our example showing that there are more general stochastic equations that still allow one to prove state vector reduction.

III. Relativistic Stochastic Equations

Because Eqs. (1) and (3) involve a universal time variable t at all spatial points, they are clearly nonrelativistic. This is evident from Eq. (2), which states that the same Itô stochastic differential is present everywhere in space, giving Wiener processes at space-like separated points that are totally correlated. In this section we explore the possibility of extending Eq. (1) into an equation with *local* Wiener processes, which can then be generalized to manifestly covariant form. We shall work henceforth with a relativistic quantum field theory, rather than with a nonrelativistic quantum mechanical system, and thus will seek to modify the Schrödinger equation for this field system to a stochastic Schrödinger equation analogous to Eq. (1), in a manner that preserves relativistic covariance. For closely related work, from which our analysis differs in some details, see Pearle [18] and Ghirardi, Grassi, and Pearle [19].

III.1 The interaction picture

Suppose we choose a particular Lorentz frame with coordinates $t, \vec{x}$, and define a state vector $|\psi\rangle$ for a field system at time t. This state evolves to a new state at time $t + dt$ according to the Schrödinger equation

$$|d\psi(t)\rangle = -i\hat{H}|\psi(t)\rangle dt \quad . \tag{25}$$

The Hamiltonian $\hat{H}$ is the integral of a Hamiltonian density $\hat{H}(\vec{x})$ over a constant time surface,

$$\hat{H} = \int d^3\vec{x}\,\hat{H}(\vec{x}) \quad . \tag{26}$$

Neither the Hamiltonian $\hat{H}$ nor the Hamiltonian density $\hat{H}(\vec{x})$ are Lorentz invariant, and the Hamiltonian densities at points $\vec{x}$ and $\vec{y}$ do not commute,

$$[\hat{H}(\vec{x}), \hat{H}(\vec{y})] = -i\vec{\nabla}_{\vec{x}}\delta^3(\vec{x}-\vec{y}) \cdot \left(\hat{\vec{P}}(\vec{x}) + \hat{\vec{P}}(\vec{y})\right) \quad , \tag{27}$$

with $\hat{\vec{P}}(\vec{x})$ the momentum density. These facts make it difficult to directly extend Eq. (25) into a stochastic Schrödinger equation in a manner consistent with Lorentz invariance.

As a first step in avoiding these problems, let us switch to the *interaction picture.* (Our use of this is heuristic and ignores mathematical issues of the existence of the interaction picture, as discussed e.g. in [22].) We write the Hamiltonian density $\hat{H}(\vec{x})$ as a sum

$$\hat{H}(\vec{x}) = \hat{H}_0(\vec{x}) + \hat{H}_{\rm int}(\vec{x}) \quad , \tag{28}$$

where $\hat{H}_0(\vec{x})$ is the free-field Hamiltonian density and $\hat{H}_{\rm int}(\vec{x})$ is the interaction Hamiltonian density. Unlike $\hat{H}(\vec{x})$ as a whole, $\hat{H}_{\rm int}(\vec{x})$ is a relativistic invariant in theories without derivative couplings, and it commutes with itself at different points,

$$[\hat{H}_{\rm int}(\vec{x}), \hat{H}_{\rm int}(\vec{y})] = 0 \quad . \tag{29}$$

Let $\hat{U}$ be the unitary time evolution operator for the free-field Hamiltonian,

$$\hat{U} = \exp\{i\hat{H}_0 t\} \quad . \tag{30}$$

If $|\psi_S(t)\rangle$ is the state at time t in the Schrödinger picture, the state in the interaction picture is $|\psi(t)\rangle = \hat{U}|\psi_S(t)\rangle$. We similarly replace the Schrödinger picture field operators (e.g., $\hat{\phi}, \hat{\pi}$) with interaction picture operators (e.g., $\hat{\phi}(t) = \hat{U}\hat{\phi}\hat{U}^\dagger, \hat{\pi}(t) = \hat{U}\hat{\pi}\hat{U}^\dagger$). If we express the interaction Hamiltonian density $\hat{H}_{\rm int}(\vec{x})$, which is a function of the field operators at the point $\vec{x}$, as a function of the interaction picture field operators, the state then obeys the simple evolution equation

$$|d\psi(t)\rangle = -i\hat{H}_{\rm int}|\psi(t)\rangle dt \quad , \tag{31}$$

where

$$\hat{H}_{\rm int} = \int d^3\vec{x}\, \hat{H}_{\rm int}(\vec{x}) \quad . \tag{32}$$

One must now remember that operators that were time independent in the Schrödinger picture acquire a time dependence governed by the free Hamiltonian $\hat{H}_0$.

So far this discussion has been restricted to constant-time surfaces in a single Lorentz frame, in which a fixed time step dt is taken simultaneously at all spatial points $\vec{x}$, and so Eq. (31) is still not Lorentz invariant. We now follow Tomonaga and Schwinger [23,24] (see also Matthews, Kroll, and Dyson [25–27]) in generalizing Eq. (32) into a *local* evolution equation. Consider a spacelike surface σ with local coordinates $\vec{x}$, on which the state of the underlying quantum fields is described by a Fock space state vector $|\psi(\sigma)\rangle$. Instead of advancing the whole spacelike surface σ, we instead move the surface forward (i.e., in the normal direction) by an increment $dt(\vec{x})$ only in the vicinity of a single point $\vec{x}$, distorting the surface σ to a new spacelike surface σ'. Under this evolution, the state vector $|\psi\rangle$ evolves to a state vector $|\psi\rangle + d_{\vec{x}}|\psi\rangle$, with the change in the state vector given by

$$d_{\vec{x}}|\psi\rangle = -i\hat{H}_{\rm int}(\vec{x})|\psi\rangle dt(\vec{x}) \quad . \tag{33}$$

The change in the state vector resulting from advancing the *entire* surface is then

$$|d\psi\rangle = \int d^3\vec{x}\, d_{\vec{x}}|\psi\rangle \quad . \tag{34}$$

Since the $\hat{H}_{\rm int}(\vec{x})$ at all points commute, the order in which the spacelike surface is advanced is immaterial and so the right hand side of Eq. (34) can be unambiguously integrated, which for constant-time surfaces with $dt(\vec{x}) \equiv dt$ recovers the original interaction picture Schrödinger equation of Eqs. (31)-(32). From the viewpoint of constructing a stochastic generalization, the local form of the interaction picture evolution equation given in Eq. (33) has three advantages: It is readily put in manifestly covariant form, it involves only the Lorentz scalar operator density $\hat{H}_{\rm int}(\vec{x})$, and this operator commutes with itself (and with other easily constructed scalar densities) at spacelike separations.

III.2 The local norm-preserving stochastic equation

Let us now replace the local unitary evolution equation of Eq. (33) with a new equation

$$d_{\vec{x}}|\psi\rangle = \hat{\alpha}(\vec{x})|\psi\rangle dt(\vec{x}) + \hat{\beta}(\vec{x})|\psi\rangle d\xi(\vec{x}) \quad , \tag{35}$$

in which we take the coefficient functions $\hat{\alpha}(\vec{x})$, $\hat{\beta}(\vec{x})$ and $\hat{\alpha}(\vec{y})$, $\hat{\beta}(\vec{y})$ to mutually commute for all $\vec{x}$, $\vec{y}$, so that no noncommutativity problems are encountered when we compound evolutions for different values of $\vec{x}$. Here $d\xi(\vec{x})$ is a complex stochastic differential variable defined at each point $\vec{x}$, which has zero stochastic mean (i.e., $M[d\xi(\vec{x})] = 0$), and which obeys the local Itô calculus

$$d\xi^*(\vec{x})d\xi(\vec{y}) = \delta^3(\vec{x}-\vec{y})dt(\vec{x}) \quad , \quad d\xi(\vec{x})d\xi(\vec{y}) = dt(\vec{x})d\xi(\vec{y}) = 0 \quad . \tag{36}$$

The spatially integrated form corresponding to Eq. (35) is

$$|d\psi\rangle = \int d^3\vec{x}\, d_{\vec{x}}|\psi\rangle = \int d^3\vec{x}[\hat{\alpha}(\vec{x})|\psi\rangle dt(\vec{x}) + \hat{\beta}(\vec{x})|\psi\rangle d\xi(\vec{x})] \quad . \tag{37}$$

In analogy with our discussion of Sec. II.1, we can now determine the conditions on the coefficient functions $\hat{\alpha}(\vec{x})$ and $\hat{\beta}(\vec{x})$ for Eq. (37) to preserve the norm of the state,

$$\begin{aligned} d\langle\psi|\psi\rangle =& \langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle + \langle d\psi|d\psi\rangle \\ =& \int d^3\vec{x}\,\langle\psi|[\hat{\alpha}(\vec{x}) + \hat{\alpha}(\vec{x})^\dagger + \hat{\beta}(\vec{x})^\dagger\hat{\beta}(\vec{x})]|\psi\rangle dt(\vec{x}) \\ +& \int d^3\vec{x}\,[\langle\psi|\hat{\beta}(\vec{x})^\dagger|\psi\rangle d\xi^*(\vec{x}) + \langle\psi|\hat{\beta}(\vec{x})|\psi\rangle d\xi(\vec{x})] \quad . \end{aligned} \tag{38}$$

Since $d\xi^*(\vec{x})$, $d\xi(\vec{x})$, and $dt(\vec{x})$ are linearly independent, the normalization of the state $\langle\psi|\psi\rangle = 1$ is preserved if and only if for all $\vec{x}$ we impose the conditions

$$\begin{aligned} 0 =& \langle\psi|[\hat{\alpha}(\vec{x}) + \hat{\alpha}(\vec{x})^\dagger + \hat{\beta}(\vec{x})^\dagger\hat{\beta}(\vec{x})]|\psi\rangle \quad , \\ 0 =& \langle\psi|\hat{\beta}(\vec{x})|\psi\rangle \quad . \end{aligned} \tag{39}$$

Evidently, if we were to replace $\hat{\alpha}$ in Sec. II.1 by $\sum_k \hat{\alpha}_k$, then Eq. (39) could be viewed as a local version of Eq. (6), with $\vec{x}$ playing the role of the index k. Imposing the normalization conditions, and specializing henceforth to $dt(\vec{x}) \equiv dt$ and flat spacelike surfaces σ, we find the following local version of Eq. (10),

$$
\begin{aligned}
d\langle\hat{O}\rangle =&\langle d\hat{O}\rangle + \int d^3\vec{x}\,[\langle\hat{\alpha}^\dagger(\vec{x})\hat{O} + \hat{O}\hat{\alpha}(\vec{x}) + \hat{\beta}^\dagger(\vec{x})\hat{O}\hat{\beta}(\vec{x})\rangle]dt \\
&+ \int d^3\vec{x}[d\xi(\vec{x})\langle\hat{O}\hat{\beta}(\vec{x})\rangle + d\xi^*(\vec{x})\langle\hat{\beta}^\dagger(\vec{x})\hat{O}\rangle] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
dV[\hat{O}] =&\langle\Delta\hat{O}d\hat{O} + d\hat{O}\Delta\hat{O}\rangle \\
&+ \int d^3\vec{x}\,[\langle\hat{\alpha}^\dagger(\vec{x})(\Delta\hat{O})^2 + (\Delta\hat{O})^2\hat{\alpha}(\vec{x}) + \hat{\beta}^\dagger(\vec{x})(\Delta\hat{O})^2\hat{\beta}(\vec{x})\rangle - 2\langle\hat{\beta}^\dagger(\vec{x})\Delta\hat{O}\rangle\langle\Delta\hat{O}\hat{\beta}(\vec{x})\rangle]dt \\
&+ \int d^3\vec{x}\,[d\xi(\vec{x})\langle(\Delta\hat{O})^2\hat{\beta}(\vec{x})\rangle + d\xi^*(\vec{x})\langle\hat{\beta}^\dagger(\vec{x})(\Delta\hat{O})^2\rangle] \quad .
\end{aligned}
\tag{40}
$$

Instead of working with the most general form of the normalization condition, we shall specialize (as we did in Sec. II.2) and satisfy Eq. (39) by taking $\hat{\alpha}(\vec{x})$ and $\hat{\beta}(\vec{x})$ to have the form

$$
\begin{aligned}
\hat{\alpha}(\vec{x}) =& - i\hat{H}_{\rm int}(\vec{x}) - \frac{1}{2}\hat{\beta}(\vec{x})^\dagger\hat{\beta}(\vec{x}) \quad , \\
\hat{\beta}(\vec{x}) =&\Delta\hat{S}(\vec{x}) \quad ,
\end{aligned}
\tag{41}
$$

with $\hat{S}(\vec{x})$ any local Lorentz scalar operator that commutes with $\hat{H}_{\rm int}(\vec{x})$. We shall further assume $\hat{S}(\vec{x})$ to be self-adjoint. Additionally, we shall assume that the operator $\hat{O}$ is self-adjoint and has no intrinsic time dependence in the Schrödinger picture, so that in the interaction picture its time dependence is given by

$$
\frac{d\hat{O}}{dt} = i[\hat{H}_0, \hat{O}] \quad . \tag{42}
$$

With these specializations, Eq. (40) can be rewritten after a little algebra as

$$
\begin{aligned}
d\langle\hat{O}\rangle =& \left[\langle i[\hat{H},\hat{O}] - \frac{1}{2}\int d^3\vec{x}\,[\hat{S}(\vec{x}),[\hat{S}(\vec{x}),\hat{O}]]\rangle\right] dt \\
&+ \int d^3\vec{x}[d\xi(\vec{x})\langle\hat{O}\Delta\hat{S}(\vec{x})\rangle + d\xi^*(\vec{x})\langle\Delta\hat{S}(\vec{x})\hat{O}\rangle] \quad ,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
dV[\hat{O}] =& \left[\langle i[\hat{H},(\Delta\hat{O})^2] - \frac{1}{2}\int d^3\vec{x}\,[\hat{S}(\vec{x}),[\hat{S}(\vec{x}),(\Delta\hat{O})^2]]\rangle - 2\int d^3\vec{x}\,|\langle\Delta\hat{O}\Delta\hat{S}(\vec{x})\rangle|^2\right] dt \\
&+ \int d^3\vec{x}\,[d\xi(\vec{x})\langle(\Delta\hat{O})^2\Delta\hat{S}(\vec{x})\rangle + d\xi^*(\vec{x})\langle\Delta\hat{S}(\vec{x})(\Delta\hat{O})^2\rangle] \quad .
\end{aligned}
$$

Although not needed for our purposes, by using the fact that $d^3\vec{x}dt$ and $\hat{S}(\vec{x})$ are Lorentz scalars, the stochastic and drift terms in Eq. (43) can be readily written in manifestly covariant form. The corresponding covariant transcription of the Hamiltonian evolution terms is given in Matthews [25] and Kroll [26].

III.3 Reduction for local density eigenstates

Let us now apply the above formulas to discuss state vector reduction to local density eigenstates, giving a relativistic generalization of the localization models discussed in [3–5]. Let us make the specific choice

$$\hat{S}(\vec{x}) = C\hat{H}_{\rm int}(\vec{x}) \quad , \tag{44}$$

which obviously satisfies the commutativity conditions

$$[\hat{\alpha}(\vec{x}),\hat{\alpha}(\vec{y})] = [\hat{\alpha}(\vec{x}),\hat{\beta}(\vec{y})] = [\hat{\beta}(\vec{x}),\hat{\beta}(\vec{y})] = [\hat{\alpha}(\vec{x}),\hat{H}_{\rm int}(\vec{y})] = [\hat{\beta}(\vec{x}),\hat{H}_{\rm int}(\vec{y})] = 0 \quad , \tag{45}$$

for all spacelike separated points $\vec{x}$, $\vec{y}$. In field theories like the Standard Model, in which all mass comes from spontaneous symmetry breaking, the mass terms arise from $\hat{H}_{\rm int}(\vec{x})$, and so for bulk matter we are effectively taking $\hat{S}$ to be the local mass density operator, multiplied by a scale factor C.

As a concrete illustration of how Eq. (43) can lead to state vector reduction and localization, let us consider the simplified case of an apparatus connected to a pointer with two macroscopic states specified by two values $\vec{X}_1$, $\vec{X}_2$ of the pointer center of mass variable $\hat{\vec{X}}$,

$$\hat{\vec{X}} \equiv \frac{\int_{\rm pointer} d^3\vec{x}\vec{x}\hat{S}(\vec{x})}{\int_{\rm pointer} d^3\vec{x}\hat{S}(\vec{x})} \quad . \tag{46}$$

We shall apply Eq. (43) to this system, taking $\hat{O} = \hat{\vec{X}}$. By Eq. (45), the double commutators $[\hat{S}(\vec{x}), [\hat{S}(\vec{x}), \hat{O}]]$ and $[\hat{S}(\vec{x}), [\hat{S}(\vec{x}), (\Delta\hat{O})^2]]$ both vanish, but in general the commutators $[\hat{H}, \hat{O}]$ and $[\hat{H}, (\Delta\hat{O})^2]$ are nonzero. However, if we take the two macroscopic pointer positions to be degenerate in energy, then the commutators involving $\hat{H}$ vanish within the degenerate two-state subspace. Taking the stochastic mean $M[\ ]$ of Eqs. (43), we then find within the two-state subspace the simplified equations

$$\begin{aligned} M[d\langle\hat{\vec{X}}\rangle] =&0 \quad , \\ M[dV[\hat{\vec{X}}]] =& -2\int d^3\vec{x} M[|\langle\Delta\hat{\vec{X}}\Delta\hat{S}(\vec{x})\rangle|^2]dt \quad . \end{aligned} \tag{47}$$

We have here exactly the same structure as we found in Eq. (19) above, and the function $f(\hat{\vec{X}}) \equiv \int d^3\vec{x}|\langle\Delta\hat{\vec{X}}\Delta\hat{S}(\vec{x})\rangle|^2$ is easily seen [c.f. the final line in Eq. (48) below] to obey the condition that the vanishing of $f(\hat{\vec{X}})$ implies the vanishing of $\langle(\Delta\hat{\vec{X}})^2\rangle$. Hence the same argument as was used in Eqs. (23) and (24) proves that an initial superposition of the two center of mass eigenstates reduces to either the state with $\hat{\vec{X}} = \vec{X}_1$ or the state with $\hat{\vec{X}} = \vec{X}_2$, with respective probabilities given by the amplitude squared to find the initial state in the respective $\hat{\vec{X}}$ eigenstate.

From Eq. (47), we can estimate the reduction rate Γ as follows. Writing $|\psi\rangle = |\vec{X}_1\rangle\cos\theta + |\vec{X}_2\rangle\sin\theta$, and assuming that the states $|\vec{X}_1\rangle$, $|\vec{X}_2\rangle$ differ sufficiently for us to approximate that $\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_2\rangle \simeq 0$, we have after a short calculation

$$\begin{aligned}
V[\hat{\vec{X}}] =&\langle(\Delta\hat{\vec{X}})^2\rangle = \sin^2\theta\cos^2\theta(\vec{X}_1-\vec{X}_2)^2 \quad , \\
\langle\Delta\hat{\vec{X}}\Delta\hat{S}(\vec{x})\rangle =&\sin^2\theta\cos^2\theta(\vec{X}_1-\vec{X}_2)(\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_1\rangle - \langle\vec{X}_2|\hat{S}(\vec{x})|\vec{X}_2\rangle) \quad , \\
|\langle\Delta\hat{\vec{X}}\Delta\hat{S}(\vec{x})\rangle|^2 =&\sin^4\theta\cos^4\theta(\vec{X}_1-\vec{X}_2)^2|\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_1\rangle - \langle\vec{X}_2|\hat{S}(\vec{x})|\vec{X}_2\rangle|^2 \\
=&\langle(\Delta\hat{\vec{X}})^2\rangle^2 \frac{|\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_1\rangle - \langle\vec{X}_2|\hat{S}(\vec{x})|\vec{X}_2\rangle|^2}{(\vec{X}_1-\vec{X}_2)^2} \quad .
\end{aligned} \tag{48}$$

Thus Eq. (47) becomes

$$\frac{M[d(\sin^2\theta\cos^2\theta)]}{dt} = -2\int d^3\vec{x}\, M[\sin^4\theta\cos^4\theta]|\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_1\rangle - \langle\vec{X}_2|\hat{S}(\vec{x})|\vec{X}_2\rangle|^2 \quad , \tag{49}$$

from which we see that, up to numerical factors of order unity, the reduction rate is given by

$$\begin{aligned}
\Gamma \sim &\int d^3\vec{x}\, |\langle\vec{X}_1|\hat{S}(\vec{x})|\vec{X}_1\rangle - \langle\vec{X}_2|\hat{S}(\vec{x})|\vec{X}_2\rangle|^2 \\
\sim &C^2\int_{\text{pointer}} d^3\vec{x}\, [\text{ Mass Density}]^2 \quad .
\end{aligned} \tag{50}$$

For a pointer containing $N \sim 10^{23}$ nucleons of mass $M \sim 1$ GeV and volume $V \sim 10^{-39}$ cm^3, the estimate of Eq. (50) becomes

$$\Gamma \sim C^2NM^2V^{-1} \quad , \tag{51}$$

which gives a reduction rate $\Gamma > 10^8$ sec^{-1} (corresponding to a collapse time faster than characteristic observational time scales) for $C > (10^9 \text{ GeV})^{-2}$. This corresponds to a mass scale at roughly the geometric mean between the Planck mass and a nucleon mass. Thus, in contrast to the energy driven model [7,12,17] for state vector reduction, where the mass scale for the coefficient of the noise terms is Planckian, in the local version discussed here the mass scale for the noise terms is much below the Planck scale, but still large compared to elementary particle masses.

III.4 Energy nonconservation

Except for the special case of stochastic equations in which the Lindblads are taken to be operators that commute with the Hamiltonian (including the Hamiltonian itself), stochastic modifications of the Schrödinger equation lead to energy nonconservation, as has been noted in the papers of Ghirardi, Rimini, and Weber, and of Pearle [1,3,18]. Let us examine this issue in the context of the relativistic model discussed above. For any operator $\hat{O}$, the stochastic expectation $M[\ ]$ of the first formula in Eq. (43) is

$$M[d\langle\hat{O}\rangle] = M\left[\langle i[\hat{H},\hat{O}] - \frac{1}{2}\int d^3\vec{x}[\hat{S}(\vec{x}),[\hat{S}(\vec{x}),\hat{O}]]\rangle\right] dt \quad , \tag{52}$$

which when applied to the Hamiltonian (i.e., taking $\hat{O} = \hat{H}$) gives for the mean rate of energy nonconservation

$$M\left[\frac{d\langle\hat{H}\rangle}{dt}\right] = -\frac{1}{2}\int d^3\vec{x}\, M[\langle[\hat{S}(\vec{x}),[\hat{S}(\vec{x}),\hat{H}]]\rangle] \quad . \tag{53}$$

In typical field theory models, the double commutator appearing in Eq. (53) is not only nonzero, but as first noted by Pearle [18] is proportional to $\delta^3(\vec{0})$ and thus is infinite. For example, taking a Dirac field model with

$$\hat{H} = \int d^3\vec{x}\, \hat{\psi}^\dagger(\vec{x})[i^{-1}\vec{\alpha}\cdot\vec{\nabla} + \beta\phi(\vec{x})]\hat{\psi}(\vec{x}) = \hat{H}_0 + \hat{H}_{\rm int} \quad , \tag{54}$$

with $\phi(\vec{x})$ an external scalar field with nonzero vacuum expectation, and choosing

$$\hat{S}(\vec{x}) = C\hat{H}_{\rm int}(\vec{x}) = C\hat{\psi}^\dagger(\vec{x})\beta\phi(\vec{x})\hat{\psi}(\vec{x}) \quad , \tag{55}$$

one has

$$[\hat{S}(\vec{x}),[\hat{S}(\vec{x}),\hat{H}]] = \delta^3(\vec{0})C^2\phi^2(\vec{x})i^{-1}[\hat{\psi}^\dagger(\vec{x})\vec{\alpha}\cdot\vec{\nabla}\hat{\psi}(\vec{x}) - \vec{\nabla}\hat{\psi}^\dagger(\vec{x})\cdot\vec{\alpha}\hat{\psi}(\vec{x})] \quad . \tag{56}$$

Similar results are found in scalar meson field theory models, and appear to be generic. Moreover, except for special choices of $\hat{S}(\vec{x})$ (see, e.g. [18-20]), the coefficient of $\delta^3(\vec{0})$ is a nontrivial operator and not a constant. The $\delta^3(\vec{0})$ singularity is a direct result of the local derivative structure of the drift term, and we have not found a mechanism to cancel it within the standard stochastic differential equation and quantum field theory framework discussed here.

IV. Conclusions

We have presented two generalizations of stochastic Schrödinger equations for state vector collapse. First, we have shown that there is an infinite parameter family of such equations for which one can prove state vector collapse with probabilities given by the Born rule. Second, we have given a relativistic stochastic equation which can be made manifestly covariant, and which produces localization onto mass density eigenstates. This produces spatial localization for superpositions of macroscopically distinct system states; to give rapid enough state vector localization in plausible experimental setups, the scale mass governing the stochastic terms must be considerably smaller than the Planck mass. The local equation has the defect that it leads to a divergent rate of energy nonconservation in generic field theory models, indicating that new ideas will be needed to achieve a satisfactory relativistic state vector collapse model.

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